ON THE CONSTRUCTION OF CERTAIN CURVES GIVEN IN POLAR COÖRDINATES.

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1. *Definitions.* Consider the parametric equations in cartesian coördinates,

$$(1) \qquad y = a \cos pt + k,$$

$$(2) \qquad x = qt.$$

(1) represents a simple harmonic motion, t the time, y the distance of the vibrating point from a fixed point in the line along which the vibration takes place, a the amplitude and $2\pi/p$ the period of the vibration, k the distance of the mean point of vibration from the fixed point. (2) represents a uniform linear (translatory) motion at right angles to the direction of the line along which the simple harmonic motion takes place, q the uniform velocity of the point, x its distance from a fixed point at any given time t. (1) and (2) taken simultaneously represent the motion resulting from the composition of these two motions. This resultant motion we shall call a *translatory-harmonic* motion, the locus of this motion the *linearly-harmonic* curve, whose equation

$$(3) \qquad y = a \cos \frac{p}{q} x + k$$

is obtained by eliminating the parameter t from the component equations (1) and (2).

Definition 1. A translatory-harmonic motion ($y = a \cos pt + k$, $x = qt$) *is the motion of a point which has simple harmonic motion* ($y = a \cos pt + k$) *along a line, while at the same time the line moves with a constant velocity* q *at right angles to itself. The locus of the resultant motion is the linearly-harmonic curve* $y = a \cos (p/q)x + k$.

Let us now consider the parametric equations in polar coördinates,

$$(4) \qquad \rho = a \cos pt + k,$$

$$(5) \qquad \theta = qt.$$

Like (1), (4) represents a simple harmonic motion, ρ being the distance of the vibrating point from a fixed point in the line along which the vibration takes place, while a, p, t and k have the same meaning as in (1). (5) represents a uniform angular (rotatory) motion, q the uniform angular velocity, θ the vectorial angle of the rotating point at any given time t. (4) and (5) taken simultaneously represent the motion resulting from the composition of these two motions. This resultant motion we shall call a *rotatory-harmonic* motion, the locus of this motion is the *cyclic-harmonic* curve

$$(6) \qquad \rho = a \cos \frac{p}{q} \theta + k,$$

obtained by eliminating the parameter t from the component equations (4) and (5).

Definition 2. A rotatory-harmonic motion ($\rho = a \cos pt + k$, $\theta = qt$) *is the motion of a point which has simple harmonic motion* ($\rho = a \cos pt + k$) *along a line, while at the same time this line rotates with a constant angular velocity q about one of its fixed points. The locus of the resultant motion is the cyclic-harmonic curve* $\rho = a \cos (p/q)\theta + k$.

2. *Construction of Cyclic-harmonic Curves by Points.* The foregoing definition furnishes a convenient method of constructing by points any cyclic-harmonic curve whose equation is given.

Let the equation of the curve be $\rho = a \cos (p/q)\theta + k$.

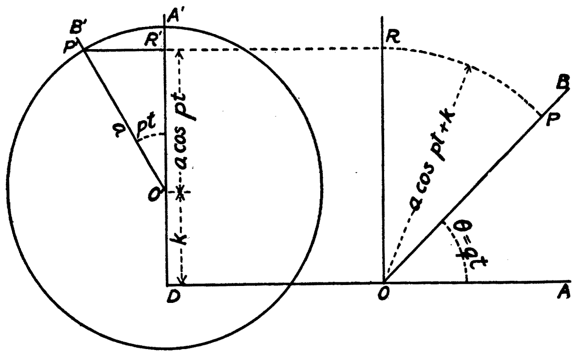

FIG. 1.

Let O (Fig. 1) represent the origin of coördinates, OA the initial line. At a point D on AO produced, chosen at any convenient distance from O, construct a perpendicular DA' and on it take DO' equal to k. Select any convenient unit of angular measure and construct the AOB ($= \theta$) and $A'O'B'$ ($= \theta'$) equal to qt and pt units respectively, t being any arbitrarily assumed integer. On $O'B'$ take $O'P'$ equal to a and let R' be the projection of P' on DA'. With O as a center and DR' as a radius describe an arc. Then P, the intersection point of this arc with OB in case DR' is positive, or with BO produced in case DR' is negative, that is in case R' falls below D, is a point on the required curve; for in either case

$$\rho = OP = OR = DR' = DO' + O'R' = k + a \cos pt, \quad \theta = qt,$$

from which on eliminating t, $\rho = a \cos (p/q)\theta + k$.

By choosing the unit of angular measure sufficiently small, and taking in turn $t = 0, 1, 2, 3$, etc., we may thus construct as many points of the curve as desired, at intervals small at will. Figs. 2, 3, 4, 5, show the method applied to the construction of the cyclic-harmonics $\rho = a \cos \frac{3}{2}\theta + k$, for the values $k = 3a$, $k = a$, $k = a/2$, $k = 0$, respectively.

3. *Classification of Cyclic-harmonic Curves.* Let the ratio p/q determine the species of the curve $\rho = a \cos (p/q)\theta + k$. An inspection of the preceding figures

discloses certain properties which are independent of the particular value of the ratio p/q employed and which are therefore common to all the species. Fig. 2 has an open center and it follows from the mode of construction, as is otherwise obvious from the form of the equation, that the curve is confined between two circles whose radii are $k - a$ and $k + a$ respectively. Fig. 3 consists of leaves which meet in cusps at the center, the axial diameter of each leaf is $k + a$. Fig. 4 consists of two sets of leaves, the axial diameters of the larger set being $k + a$, those of the smaller set $k - a$. Fig. 5 consists of a single set of equal leaves whose axial diameters equal a.

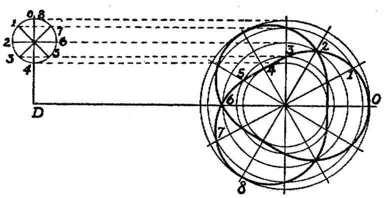

FIG. 2. Curtate Cyclic-harmonic, $\frac{p}{q} = \frac{3}{2}$, $\frac{k}{a} = 3$.

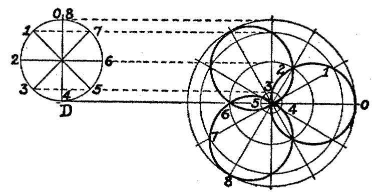

FIG. 3. Cuspitate Cyclic-harmonic $\frac{p}{q} = \frac{3}{2}$, $\frac{k}{a} = 1$.

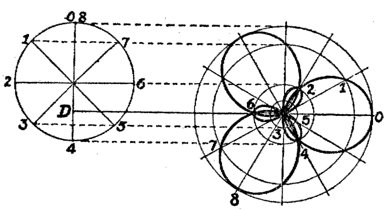

FIG. 4. Prolate Cyclic-harmonic, $\frac{p}{q} = \frac{3}{2}$, $\frac{k}{a} = \frac{1}{2}$.

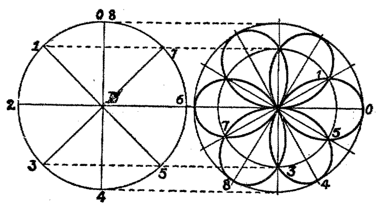

FIG. 5. Foliate Cyclic-harmonic, $\frac{p}{q} = \frac{3}{2}$, $\frac{k}{a} = 0$.

These properties serve as a convenient basis for the further classification of the cyclic-harmonic curves of any species. We shall call a cyclic-harmonic curtate if $k > a$, cuspitate if $k = a$, prolate if $0 < k < a$, equi-foliate (or foliate) if $k = 0$.

The cyclic-harmonic curves embrace as special cases a considerable number of familiar curves. Pascal's conchoids constitute one species, $p/q = 1$. The cardioid and Münger's double egg curves are cuspitate cyclic-harmonics, $p/q = 1$ and 2 respectively. The common limaçon and Freeth's nephroid are prolate cyclic-harmonics, $p/q = 1$ and $\frac{1}{2}$ respectively. All roses (Rosenkurven, rosaces) are equi-foliate cyclic-harmonics. The linearly harmonic (simple harmonic) curves will be shown to be degenerate forms of curtate cyclic-harmonic curves.

4. *The Cyclo-harmonograph.* The foregoing definition of rotatory harmonic motion suggests a simple mechanism for constructing any cyclic-harmonic curve kinematically.

A wheel W_1, center C, carries a crank-pin R which, as the wheel rotates, slides in a slotted cross-bar ST of a cross-head HK perpendicular to ST. This cross-head is constrained to move in the direction of HK by means of two fixed guides F and G. As the wheel W_1 moves with constant angular velocity about its center C, any point P in HK will have simple harmonic motion in the direction of HK.

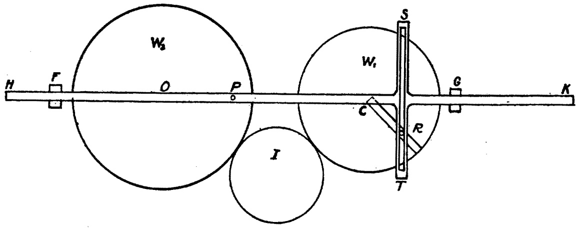

FIG. 6.

Let now the mechanism thus far described be made to revolve about a fixed wheel W_2, center O, by means of an idle-wheel I which connects the circumferences of the wheels W_1 and W_2. Any point P in HK will then receive rotatory harmonic motion and a pencil or pen-point placed at P will trace out a cyclic-harmonic curve in the plane of the paper.

To deduce the equation of the curve traced out by P, let the dotted lines in Fig. 7 represent the initial position of the mechanism, which is so chosen that the crank-pin R_0 is in the line of centers OC_0 of the wheels W_2 and W_1 and such that C_0 is between O and R_0.

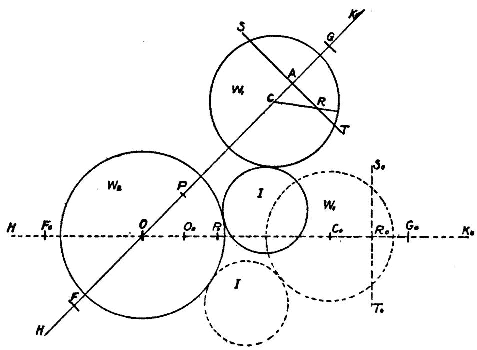

FIG. 7.

Take $R_0O_0 = C_0O$, the distance between the centers of the two wheels W_1 and W_2, and let k denote the distance of any point P_0 on H_0K_0 from O_0.

Let C represent the center of the wheel W_1 in any other position, P R H K F G S T the corresponding positions of P_0 R_0 H_0 K_0 F_0 G_0 S_0 T_0 respectively, and A the intersection of ST with HK.

Take O for the origin of coördinates, OK_0 for the initial line, and denote the coördinates of the point P by ρ, θ. Furthermore let $C_0R_0 = CR = a$, and

angle $RCK = \phi$. Then

$$OP = \rho = OC + CA - PA, \quad OC = OC_0 = O_0R_0,$$

$$CA = CR \cos\phi = a \cos\phi, \quad PA = P_0R_0 = O_0R_0 - k.$$

Hence

$$OP = \rho = O_0R_0 + a \cos\phi - (O_0R_0 - k), \text{ or } \rho = a \cos\phi + k.$$

Now let p and q represent the radii of the wheels W_2 and W_1 respectively, then obviously $q\phi = p\theta$, $\phi = (p/q)\theta$, and we have

$$\rho = a \cos\frac{p}{q}\theta + k \tag{7}$$

as the general equation of the locus of the point P.

The form of equation (7) shows that the locus of P is independent of the distance between the centers of the wheels W_1 and W_2, but depends only on the dimensions of the two wheels, the distance of the crank-pin R from the center C and the arbitrary distance O_0R_0. The single wheels W_1 and W_2 may therefore be replaced by two trains of wheels of various diameters, the idle-wheel I being used to connect at will any wheel of either train with any wheel of the other. A sliding carriage on HK and a crank-pin adjustable to various distances CR makes it possible to assign to k and c arbitrary values within the physical limits of the mechanism. Finally, it is immaterial whether the wheel W_2 is kept fixed while W_1 revolves about it, or whether the centers of both wheels remain fixed and both wheels be allowed to revolve, the paper on which the curve is traced being attached to the face of the wheel W_2. For practical reasons the latter is the more advantageous arrangement.

5. *Linearly-harmonic Curves as Degenerate Forms of Cyclic-harmonics.* Fig. 8 shows a portion of a curtate cyclic-harmonic curve. Plainly, as the arc which

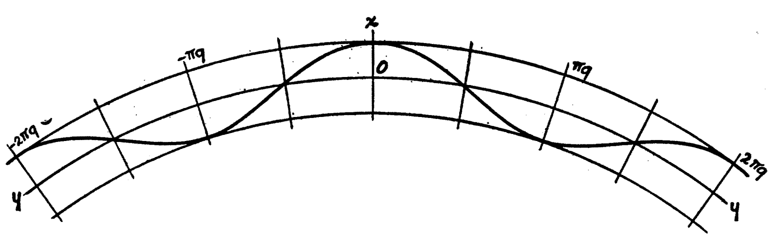

FIG. 8.

forms the directrix of this curve degenerates into a straight line the curve itself will degenerate into a linearly harmonic curve.

In Fig. 7 let x, y denote the rectangular coördinates of any point ρ, θ of the cyclic-harmonic curve $\rho = a \cos (p/q)\theta + k$, referred to C_0 as a new origin with C_0K_0 as the positive direction of the new x-axis. Put $k = l = OC_0$, the distance between the centers of the wheels W_2 and W_1. We then have

$$x + l = \rho \cos \theta, \qquad y = \rho \sin \theta, \qquad \rho = \sqrt{(x + l)^2 + y^2},$$

$$\frac{p}{q}\theta = \cos^{-1}\frac{\rho - l}{a} = \frac{p}{q}\tan^{-1}\frac{y}{x + l}. \tag{8}$$

Now let p increase indefinitely while q remains constant, then since $l = p + q + a$ constant, l also will increase indefinitely, and it is easy to see that

$$\lim_{l \doteq \infty} \frac{\rho - l}{a} = \frac{x}{a}, \qquad \lim_{l \doteq \infty} \left[\frac{p}{q}\tan^{-1}\frac{y}{x + l}\right] = \frac{y}{q},$$

hence equation (8) becomes in the limit

$$\cos^{-1}\frac{x}{a} = \frac{y}{q},$$

or

$$x = a \cos\frac{y}{q},$$

the equation of a linearly-harmonic curve, amplitude a, wave-length $2\pi q$.

6. *The Division of the Circle into Any Number of Parts.* The cyclo-harmonograph effects a complete solution of the classic problem of cyclotomy by determining the vertices of a regular polygon of any odd number of sides. It is easily shown that the cyclic harmonic curve $\rho = a \cos (p/q)\theta + k$ has a set of $p(q - 1)$ nodes corresponding to the angles

$$\theta_{\lambda, \mu} = \left(\lambda + \frac{q}{p}\mu\right)\pi, \qquad \lambda = 1, 2, 3, \cdots, q - 1; \qquad \mu = 0, 1, 2, \cdots, p - 1.$$

These nodes lie in sets of $q - 1$ on the p straight lines $\theta_\lambda = (\lambda/p)\pi$, in which $\lambda = 0, 1, 2, \cdots, p - 1$, and in sets of p on the $q - 1$ circles $\rho_\mu = a \cos (\mu/q)\pi + k$, in which $\mu = 1, 2, 3, \cdots, q - 1$. It is clear therefore that any set of these nodes situated on the same circle determines the vertices of a regular polygon with n sides. In particular the cyclic-harmonic $\rho = a \cos (p/2)\pi + k$ has the p nodes $\theta_\lambda = (\lambda/p)\pi$, in which $\rho_\lambda = k$, $\lambda = 1, 3, 5, \cdots, 2p - 1$.

7. *The Number of Species.* The number of species of cyclic-harmonics which it is possible to describe with a given cyclo-harmonograph depends of course on the number of wheels in the train of gears employed in the mechanism. Suppose that there are n wheels in each train and that the diameters of these wheels are proportional to the numbers $1, 2, 3, \cdots, n$. Let the diameters of the two wheels which the idle-wheel connects be as p is to q. Assume p constant and greater than q, with this assumption there are $\phi(p)$ admissible values of the ratio p/q, $\phi(p)$ being the totient function of p, that is, the number of numbers which are less than p and have with it no common divisor other than unity. Now let p take all values from 1 to n and we obtain $\Sigma_1^n \phi(n)$ species of curves under the restriction $n \geqq p > q$. Evidently there is an equal number of species under the restriction $n \geqq q > p$. Besides these there is the case $p/q = 1$. The total

number of species within the range of a cyclo-harmonograph having n wheels in each train of gears is therefore $1 + 2\,\Sigma_1^n\,\phi(n)$.

8. *Explanation of Plates.* The plates which follow contain four each of six species of cyclo-harmonic curves corresponding to the values $p/q = 2,\ 1/2,\ 9/2,\ 2/9,\ 8/5$, and $10/9$ respectively. Of each set of four, the first is curtate, the second cuspitate, the third prolate, and the fourth equi-foliate. The pole is taken at the center of each figure, the polar axis extending to the right. In the cases $k = 0$, k has actually been chosen slightly different from zero in order to exhibit more clearly the cusps at the origin.

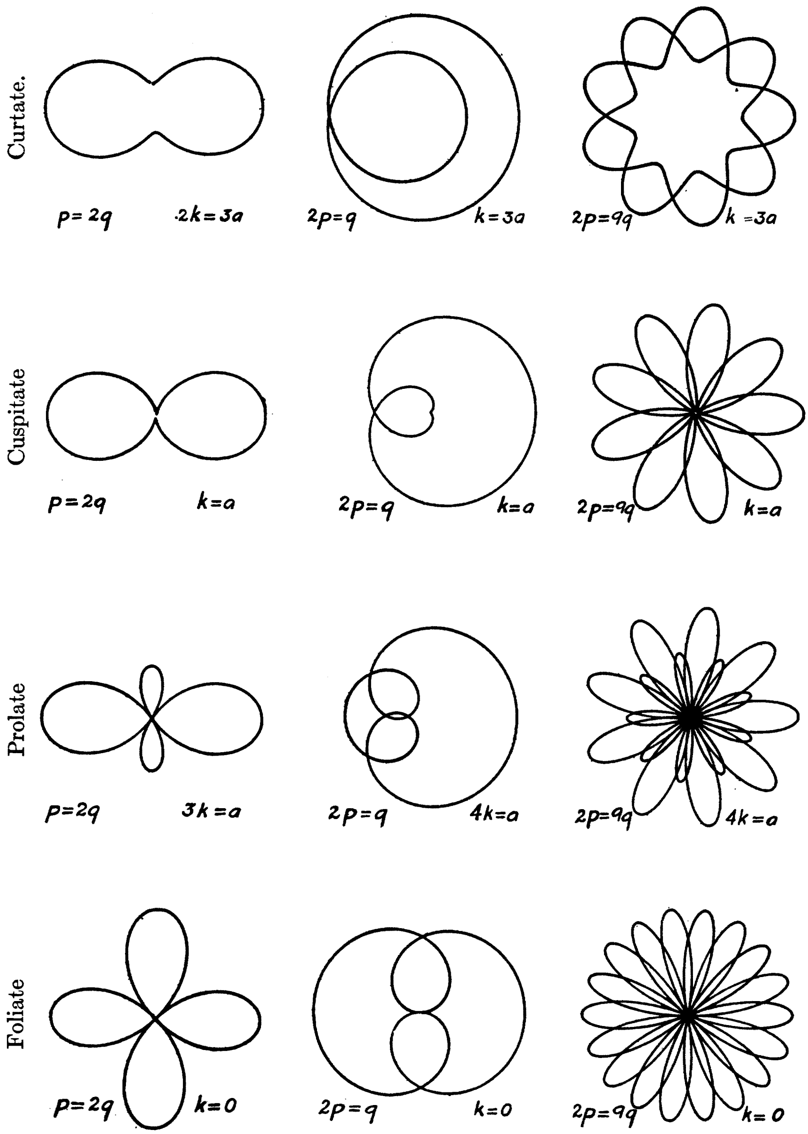

Plate I.

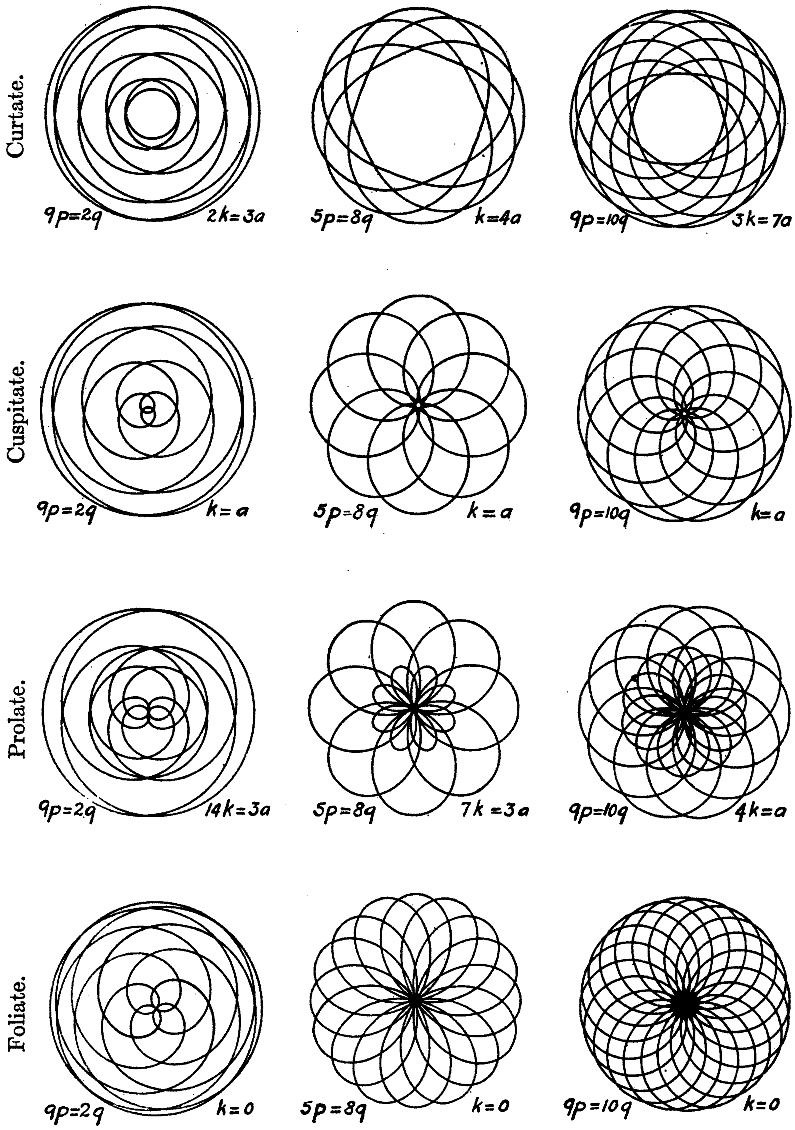
Curtate.
9p=2q
2k=3a
5p=8q
k=4a
9p=10q
3k=7a
Cuspitate.
9p=2q
k=a
5p=8q
k=a
9p=10q
k=a
Prolate.
9p=2q
14k=3a
5p=8q
7k=3a
9p=10q
4k=a
Foliate.
9p=2q
k=0
5p=8q
k=0
9p=10q
k=0

PLATE II.